

Odd Hole Recognition in Graphs of Bounded Clique Size

Michele Conforti

Dipartimento di Matematica Pura ed Applicata

Università di Padova

Via Belzoni 7, 35131 Padova, Italy

conforti@math.unipd.it

Gérard Cornuéjols ¹

Tepper School of Business

Carnegie Mellon University

Pittsburgh, PA 15213 and

LIF, Faculté des Sciences de Luminy

13288 Marseille, France

gc0v@andrew.cmu.edu

Xinming Liu

Tepper School of Business

Carnegie Mellon University

Pittsburgh, PA 15213

xinming.liu@gmail.com

Kristina Vušković ²

School of Computing

University of Leeds

Leeds LS2 9JT, UK

vuskovi@comp.leeds.ac.uk

Giacomo Zambelli

Department of Combinatorics and Optimization

University of Waterloo

Waterloo, Ontario N2L 3G1, Canada

gzambell@math.uwaterloo.ca

July 19, 2004

Revised August 23, 2005

¹Supported by NSF grant DMI-0352885 and ONR grant N00014-03-1-0133.

²Supported by EPSRC grant GR/R35629/01.

Abstract

In a graph G , an odd hole is an induced odd cycle of length at least five. A clique of G is a set of pairwise adjacent vertices. In this paper we consider the class \mathcal{C}_k of graphs whose cliques have a size bounded by a constant k . Given a graph G in \mathcal{C}_k , we show how to recognize in polynomial time whether G contains an odd hole.

Keywords: odd hole, recognition algorithm, cleaning, decomposition

1 Introduction

A *hole* is a graph induced by a cycle of length at least four. A hole is *odd* if it contains an odd number of vertices. Otherwise, it is even. Graph G *contains* graph H if H is isomorphic to an induced subgraph of G . Chudnovsky, Cornu  jols, Liu, Seymour and Vu  kovi   recently proved that it is polynomial to test whether a graph contains an odd hole or its complement [2]. However, it is still an open problem to test whether a graph contains an odd hole. Bienstock [1] proved that it is *NP*-complete to test whether a graph contains an odd hole passing through a specific vertex. A *clique* is a set of pairwise adjacent vertices. The *clique number* of a graph is the size of its largest clique. In this paper, we show that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

We use the same general strategy as in [2]. Let H be an odd hole in a graph G . We say that $u \in V(G) \setminus V(H)$ is *H-minor* if its neighbors in H lie in some 2-edge path of H . In particular, u is *H-minor* if u has no neighbor in H . A vertex $u \in V(G) \setminus V(H)$ is *H-major* if it is not *H-minor*. We say that H is *clean* if G contains no *H-major* vertex. A graph G is *clean* if either it is odd-hole-free or it contains a clean shortest odd hole. As in [2] our approach for testing whether a graph G of bounded clique number contains an odd hole consists of two steps:

- (i) constructing in polynomial time a clean graph G' that contains an odd hole if and only if G does, or in some cases identifying an odd hole of G , and
- (ii) checking whether the clean graph G' contains an odd hole.

For step (ii), we can use the polynomial algorithms in [2]. The main result of this paper is a polynomial algorithm for step (i). Step (i) is called *cleaning* the graph G .

1.1 Notation

For a graph G and a set B of vertices of G , we denote by $G(B)$ the subgraph of G induced by the vertex set B . For a vertex v , $N(v)$ denotes the set of vertices adjacent to v .

A *pyramid* $\Pi(xyz; u)$ is a graph induced by three paths $P_1 = x, \dots, u$, $P_2 = y, \dots, u$ and $P_3 = z, \dots, u$ having no common or adjacent intermediate vertices, such that at most one of the paths is of length 1 and the vertex set $\{x, y, z\}$ induces a clique of size 3. Note that every two of the paths P_1, P_2, P_3 induce a hole. Since two of the three paths must have the same parity, one of these holes is odd. Therefore, every pyramid contains an odd hole.

A *wheel*, denoted by (H, x) , is a graph induced by a hole H and a vertex $x \notin V(H)$ having at least three neighbors in H , say x_1, \dots, x_n . Vertex x is the *center* of the wheel. A subpath of H connecting x_i and x_j is a *sector* if it contains no intermediate vertex x_l , $l \in \{1, \dots, n\}$. A *short sector* is a sector of length 1, and a *long sector* is a sector of length at least 2. A wheel is *odd* if it contains an odd number of short sectors, and *even* otherwise. Each of the long sectors together with vertex x induces a hole. If each of these holes is even and the wheel (H, v) is odd then H is an odd hole, since the wheel (H, x) contains an odd number of short sectors. Therefore, every odd wheel contains an odd hole.

In a graph G , a *jewel* is a sequence v_1, \dots, v_5, P such that v_1, \dots, v_5 are distinct vertices, $v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1$ are edges, v_1v_3, v_2v_4, v_1v_4 are nonedges, and P is a path of G between v_1

and v_4 such that v_2, v_3, v_5 have no neighbors in $V(P) \setminus \{v_1, v_4\}$. Clearly a jewel either contains an odd wheel or a 5-hole, so if there is a jewel in a graph G then there is an odd hole in G .

Chudnovsky and Seymour found an $O(|V(G)|^9)$ algorithm to test whether a graph G contains a pyramid and an $O(|V(G)|^6)$ algorithm to test whether a graph G contains a jewel (see [2]).

2 Cleaning

In this section, we show how to clean a graph G of bounded clique number. That is, we perform step (i) above. The cleaning algorithm produces a polynomial family of induced subgraphs of G such that if G contains a shortest odd hole H^* , then one of the graphs produced by the cleaning algorithm, say G' , contains H^* and H^* is clean in G' .

Roughly speaking, this is accomplished by showing that there exists a set X of vertices of H^* , whose size depends only on the clique number, such that every major vertex for H^* has a neighbor in X . Since the set Y of vertices of H^* with neighbors in X has at most $2|X|$ elements, we may enumerate all possible choices for X and Y , and for each choice of X and Y add to the family the graph obtained by removing the vertices of $V(G) \setminus Y$ that have a neighbor in X .

2.1 Vertices with At Most Three Neighbors in H^*

Lemma 1 *Let H^* be a shortest odd hole in G . Suppose that G does not contain a pyramid. If a vertex $u \notin V(H^*)$ has a neighbor but no more than three neighbors in H^* then u is H^* -minor.*

Proof: If u has one neighbor in H^* then u is H^* -minor. Now suppose that u has two neighbors in H^* , say u_1 and u_2 . Let P_1 and P_2 be the two u_1u_2 -subpaths of H^* . Since H^* is odd, P_1 and P_2 have different parity, say P_1 is odd. If P_1 is of length 1 then u is H^* -minor. Otherwise, $V(P_1) \cup \{u\}$ induces an odd hole. Since this hole cannot be shorter than H^* , P_2 is of length 2, and hence u is H^* -minor.

Now assume that u has three neighbors in H^* , and let P_1, P_2 and P_3 be the three sectors of the wheel (H^*, u) . If exactly one of the sectors is short then $V(H^*) \cup \{u\}$ induces a pyramid. If two of the sectors are short then u is H^* -minor. Finally suppose that all three sectors are long. Since H^* is odd, at least one of the sectors, say P_1 , is odd. Then $V(P_1) \cup \{u\}$ induces an odd hole shorter than H^* , a contradiction. \square

2.2 Vertices with More Than Three Neighbors in H^*

Let H^* be a shortest odd hole in G . Let $S(H^*)$ be the set of H^* -major vertices that have four or more neighbors in H^* . Note that, for any $u \in S(H^*)$, every long sector of the wheel (H^*, u) is of even length since H^* is a shortest odd hole of G ; hence, (H^*, u) contains an odd number of short sectors.

Let $S \subseteq V(G)$. We say that vertex $x \in V(G) \setminus S$ is S -complete if x is adjacent to every vertex in S . We say that an edge xy is S -complete if both vertices x and y are S -complete.

Lemma 2 *Let H^* be a shortest odd hole in G . Suppose that G does not contain a jewel. If $u, v \in S(H^*)$ are not adjacent then an odd number of edges of H^* are $\{u, v\}$ -complete.*

Proof: Let u and v be nonadjacent vertices of $S(H^*)$. Suppose that an even number of edges of H^* are $\{u, v\}$ -complete. Then some long sector P of the wheel (H^*, u) contains an odd number of short sectors of (H^*, v) . Let u_1 and u_2 be the endvertices of P . P has even length. Let P' be the subpath of H^* induced by $(V(H^*) \setminus V(P)) \cup \{u_1, u_2\}$. P' has odd length. Note that P' must be of length at least four, since otherwise (H^*, u) is a jewel, a contradiction. If P contains three or more neighbors of v , then the vertex set $V(P) \cup \{u, v\}$ induces an odd wheel with center v , and hence contains an odd hole shorter than H^* , contradicting our choice of H^* . Otherwise, let v_1 and v_2 be the two neighbors of v in P . Vertex v cannot have exactly four neighbors in H^* , say v_1, v_2, v_3, v_4 , such that both v_3u_1 and v_4u_2 are edges, because otherwise the vertex set $(V(H^*) \setminus V(P)) \cup \{v\}$ induces a shorter odd hole than H^* , since P is even and P' is of length at least four. Therefore, there exist vertices $u_3, v_3 \in V(H^*) \setminus V(P)$, the neighbors of u and v respectively, such that u and v have no other neighbors on u_3v_3 -subpath of H^* , call it Q , and vertices u_3 and v_3 are not adjacent to u_1 or u_2 . But now the vertex set $V(Q) \cup V(P) \cup \{u, v\}$ induces a pyramid $\Pi(v_1v_2v; u)$, and hence contains an odd hole shorter than H^* , contradicting our choice of H^* . \square

The following, which is an easy consequence of Lemma 2, will be used in several places.

Lemma 3 *Let H^* be a shortest odd hole in G , P be a subpath of H^* such that $|V(H^*) \setminus V(P)| \geq 3$, and x, y be two nonadjacent vertices in $S(H^*)$. Assume that no ends of P are $\{x, y\}$ -complete and there is no $\{x, y\}$ -complete edge in P . Then there exists an $\{x, y\}$ -complete vertex in H^* with no neighbor in P .*

Proof: By Lemma 2, there exists an $\{x, y\}$ complete edge e in H^* . One of the two endvertices of e has the desired property. \square

Lemma 4 *Suppose that G does not contain a jewel. If $A \subseteq S(H^*)$ is a stable set, then an odd number of edges of H^* are A -complete.*

Proof: Let $A \subseteq S(H^*)$ be a stable set and suppose that an even number of edges of H^* are A -complete. Let A' be a smallest subset of A with the property that an even number of edges of H^* are A' -complete. Note that by Lemma 2, $|A'| \geq 3$. Let s_1, \dots, s_m be the vertices of H^* adjacent to at least one vertex in A' , encountered in that order when traversing H^* clockwise. For $i \in [m]$, let S_i be the $s_i s_{i+1}$ -subpath of H^* (indices taken modulo m), that does not contain any intermediate vertex s_j , $j \in [m]$.

Claim For every $i \in [m]$, S_i is either an edge whose endvertices are both adjacent to some vertex $x \in A$, or S_i has even length.

Proof of Claim: If there is a vertex $x \in A'$ adjacent to both s_i and s_{i+1} , then S_i is a sector of the wheel (H^*, x) and hence the result holds. Otherwise, let x_1 and x_2 be vertices of A' such that x_1 is adjacent to s_i and x_2 is adjacent to s_{i+1} . By Lemma 3 there exists an $\{x_1, x_2\}$ -complete vertex u in H^* with no neighbor in S_i . Then the vertex set $V(S_i) \cup \{x_1, x_2, u\}$ induces a hole. Since both x_1 and x_2 have at least four neighbors in H^* , this hole is shorter than H^* , so it must be even, hence S_i is of even length. This completes the proof of the claim. \diamond

For $C \subseteq A'$, let δ_C denote the number of edges of H^* that are C -complete. Let δ be the number of paths in S_1, \dots, S_m of length one. Then

$$\delta = \sum_{i=1}^{|A'|} (-1)^{i+1} \sum_{C \subseteq A', |C|=i} \delta_C$$

By the choice of A' , for every $C \subseteq A'$ such that $C \neq A'$, δ_C is odd. Hence the parity of δ is equal to the parity of

$$\sum_{i=1}^{|A'|-1} \binom{|A'|}{i} + \delta_{A'}$$

which is itself equal to the parity of $\delta_{A'}$ since

$$\sum_{i=1}^{|A'|-1} \binom{|A'|}{i} = 2^{|A'|} - 2$$

By the Claim and because H^* is odd, δ is odd. Hence $\delta_{A'}$ must be odd as well, contradicting the choice of A' . \square

Theorem 5 Suppose that G does not contain a jewel. Let A be a stable set of $S(H^*)$ and let x_1x_2 be an edge of H^* such that every vertex of A is adjacent to both x_1 and x_2 (such an edge exists by Lemma 4). Let B be the set of vertices of $S(H^*)$ that have no neighbor in $\{x_1, x_2\}$, and have both a neighbor and a nonneighbor in A . Then there exists an edge y_1y_2 of H^* such that y_1 is A -complete and every vertex of B has a neighbor in $\{y_1, y_2\}$.

Proof: If $B = \emptyset$ then the result is trivially true, so we may assume that $B \neq \emptyset$. Since every vertex of B is major, this implies that H^* is of length greater than 5.

Claim 1 For every $u \in B$, an edge of H^* is $(A \cup \{u\})$ -complete.

Proof of Claim 1: Let A_1 be the neighbors of u in A and $A_2 = A \setminus A_1$. By Lemma 4, there is an edge u_1u_2 of H^* such that every vertex of $A_2 \cup \{u\}$ is adjacent to both u_1 and u_2 . Since u has no neighbor in $\{x_1, x_2\}$, every vertex of A_1 must be adjacent to both u_1 and u_2 , else there is a 5-hole. This completes the proof of Claim 1. \diamond

Claim 2 If X is a stable set of B , then there exists an edge z_1z_2 of H^* such that z_1 is A -complete and every vertex of X has a neighbor in $\{z_1, z_2\}$.

Proof of Claim 2: We consider the following two cases.

Case 1 There is a vertex in A that is not adjacent to any vertex in X .

Let $A_1 \subseteq A$ be such that $A_1 \cup X$ is a maximal stable set. By Lemma 4, an edge of H^* is $(A_1 \cup X)$ -complete, say u_1u_2 . Let $w \in A \setminus A_1$. Note that w is adjacent to some $x \in X$. If w is not adjacent to u_1 or u_2 , then there is a 5-hole in the graph induced by $\{x, y, w, u_1, u_2, x_1, x_2\}$, where $y \in A_1$. So every vertex of $A \setminus A_1$ is adjacent to both u_1 and u_2 .

Case 2 Every vertex of A is adjacent to some vertex in X .

By Claim 1 and Case 1, we may assume w.l.o.g. that $|X| > 1$ and for every proper subset of X the result holds. Let $w \in A$ be such that $|N(w) \cap X|$ is minimum. Let $Z = N(w) \cap X$. Since every vertex of X has a non-neighbor in A and $|Z|$ is minimum, $|Z| < |X|$. By our assumption, there exists an edge y_1y_2 of H^* such that y_1 is A -complete and every vertex of $X \setminus Z$ has a neighbor in $\{y_1, y_2\}$. By Lemma 4 an edge of H^* is X -complete, say edge y_3y_4 .

We may assume that vertices y_1, y_2, y_3, y_4 are all distinct and y_1y_3 and y_1y_4 are not edges, since otherwise the result trivially holds. Also w.l.o.g. y_2y_4 is not an edge.

Suppose that wy_4 is not an edge. We may assume that some $z \in Z$ is not adjacent to y_1 , since otherwise the edge y_1y_2 satisfies the claim. If some $v \in X \setminus Z$ is adjacent to y_1 , then $\{y_1, v, w, z, y_4\}$ induces a 5-hole. So for every $v \in X \setminus Z$, vy_1 is not an edge, and hence vy_2 is an edge. If w is adjacent to y_2 , then $\{y_2, w, v, z, y_4\}$ induces a 5-hole. So w is not adjacent to y_2 . By Lemma 3, there is a vertex u of H^* adjacent to both v and w , but with no neighbor in $\{y_1, y_2\}$. Then $\{y_1, y_2, u, v, w\}$ induces a 5-hole.

Therefore wy_4 is an edge. We now show that y_4 is A -complete. Let $w' \in A$ and assume $w'y_4$ is not an edge. By the choice of w and by the above argument, there is a vertex $v \in X \setminus Z$ adjacent to w' . But then the graph induced by $\{w, w', x_1, x_2, v, y_4\}$ contains a 5-hole. This completes the proof of Claim 2. \diamond

Claim 3 For every edge v_1v_2 in $G(B)$, there exists $v \in A$ that is adjacent to neither v_1 nor v_2 .

Proof of Claim 3: Let A_1 be the set of neighbors of v_1 in A , and $A_2 = A \setminus A_1$. Suppose the claim does not hold. Then v_2 is universal for A_2 . Let w_1 be a vertex of A_1 that v_2 is not adjacent to. Then $v_1, v_2, w_2, x_2, w_1, v_1$, where $w_2 \in A_2$, is a 5-hole. This completes the proof of Claim 3. \diamond

By Claim 1, we may assume that for every proper subset B' of B , the statement holds. By Claim 2 we may assume that B is not a stable set. Let v_1v_2 be an edge of $G(B)$. By Claim 3, let v be a vertex of A that is adjacent to neither v_1 nor v_2 . Let y_1y_2 be an edge of H^* such that y_1 is A -complete and all vertices of $B \setminus v_2$ have a neighbor in $\{y_1, y_2\}$. Let y_3y_4 be an edge of H^* such that y_3 is A -complete and all vertices of $B \setminus v_1$ have a neighbor in $\{y_3, y_4\}$. Then the theorem follows from the following claim.

Claim 4 v_1 has a neighbor in $\{y_3, y_4\}$, or v_2 has a neighbor in $\{y_1, y_2\}$.

Proof of Claim 4: Suppose the claim does not hold. v_1 has no neighbor in $\{y_3, y_4\}$ and v_2 has no neighbor in $\{y_1, y_2\}$.

If a vertex of $\{y_1, y_2\}$ coincides with a vertex of $\{y_3, y_4\}$, then $\{y_1, y_2, y_3, y_4, v_1, v_2\}$ induces a 5-hole. Therefore, vertices y_1, y_2, y_3, y_4 are all distinct.

We now show that v and v_1 must have a common neighbor in $\{y_1, y_2\}$. Assume not. Then vy_1 and v_1y_2 are edges, and vy_2 and v_1y_1 are not. By Lemma 3, there is a vertex u of H^* that is $\{v, v_1\}$ -complete but has no neighbor in $\{y_1, y_2\}$. Then $\{y_1, y_2, v, v_1, u\}$ induces a 5-hole. Therefore, v and v_1 have a common neighbor y in $\{y_1, y_2\}$, and similarly v and v_2 have a common neighbor y' in $\{y_3, y_4\}$. If yy' is not an edge, then $\{y, y', v, v_1, v_2\}$ induces a 5-hole. Therefore, yy' is an edge.

Let a, y, y', b be the subpath of H^* induced by $\{y_1, y_2, y_3, y_4\}$. Then vy, vy', v_1y, v_2y' are edges and v_2a, v_2y, v_1y', v_1b are not.

Let z_2 be the neighbor of v_2 in H^* that is closest to a in $H^* \setminus \{y, y'\}$. Note that $z_2 \neq b$ since v_2 is a major vertex. Let P_2 be the az_2 -subpath of H^* that does not contain y .

Suppose v does not have a neighbor in P_2 . By Lemma 3, some vertex u of H^* is $\{v, v_2\}$ -complete and has no neighbor in P_2 . Note that $u \neq b$ since b is not $\{v, v_2\}$ -complete. But then $P_2 \cup \{y, y', v, v_2, u\}$ induces a pyramid $\Pi(vyy', v_2)$, and hence there is an odd hole shorter than H^* , a contradiction. Therefore v must have a neighbor in P_2 .

We now show that a is the unique neighbor of v in P_2 . Let v' be the neighbor of v in P_2 that is closest to z_2 . Assume that $v' \neq a$. Let P' be the $v'z_2$ -subpath of P_2 . If v_1 has no neighbor in P' , then the graph induced by $S = P' \cup \{y, y', v, v_1, v_2\}$ is a pyramid $\Pi(vyy', v_2)$ hence there is an odd hole shorter than H^* . If v_1 has a neighbor in $P' \setminus z_2$, then the graph induced by S contains a pyramid $\Pi(vyy', v_1)$ hence there is an odd hole shorter than H^* . So v_1 is adjacent to z_2 . If the graph induced by $P_2 \cup \{y, y', v_1, v_2\}$ is an odd wheel with center v_1 , there is an odd hole shorter than H^* . Hence v_1 must have a neighbor in $P_2 \setminus P'$. If v_1 has a neighbor z in P_2 that lies strictly between a and v' , then there is a path Q from v to v_1 with interior in z, P_2, v' . But then $Q \cup \{y, y', v_2\}$ induces a pyramid $\Pi(vyy', v_1)$, which contains an odd hole shorter than H^* . Therefore a and z_2 are the only neighbors of v_1 in P_2 . Then v is not adjacent to a for otherwise a, v, y', v_2, v_1, a is an odd hole. Let v'' be the neighbor of v closest to a in P_2 . Note that $v'' \neq z_2$ since otherwise $P_2 \cup \{y, y', v_2, v\}$ induces an odd wheel with center v hence there is an odd hole shorter than H^* . Let P'' denote the av'' -subpath of P_2 . By Lemma 3, some vertex u of H^* is $\{v, v_1\}$ -complete and has no neighbor in P'' . But then the graph induced by $P'' \cup \{y, v, v_1, u\}$ is a pyramid $\Pi(ayv_1, v)$ hence there is an odd hole shorter than H^* . Therefore a is the unique neighbor of v in P_2 .

Then v_1 is not adjacent to a for otherwise a, v, y', v_2, v_1, a is an odd hole. Suppose v_1 has a neighbor in P_2 . By Lemma 3, there exists a vertex u of H^* adjacent to both v and v_1 , but with no neighbor in P_2 . Then the graph induced by $P_2 \cup \{y, v, v_1, u\}$ contains a pyramid $\Pi(ayv, v_1)$ hence there is an odd hole shorter than H^* . Therefore, v_1 has no neighbor in P_2 .

Let z_1 be the neighbor of v_1 in H^* that is closest to b in $H^* \setminus \{y, y'\}$. Let P_1 be the bz_1 -subpath of H^* that does not contain y . By symmetry, b is the unique neighbor of v in P_1 and v_2 has no neighbor in P_1 . Since P_2, a, y, y' is a sector of wheel (H^*, v_2) , P_2 must be even, and similarly P_1 is even. Note that z_1z_2 is not an edge since H^* and the path a, y, y', b have odd length and P_1, P_2 have even length. But then $P_1 \cup P_2 \cup \{v, v_1, v_2\}$ induces an odd hole shorter than H^* , a contradiction. \square

2.3 Cleaning Algorithm

In this section, we present our cleaning algorithm for the class of graphs of bounded clique number. The running time depends on the clique number.

Input: A graph G of bounded clique number k .

Output: Either an odd hole or a family \mathcal{F} of induced subgraphs of G that satisfies the following properties:

- (1) G contains an odd hole if and only if some graph of \mathcal{F} contains a clean shortest odd hole.
- (2) $|\mathcal{F}|$ is $O(|V(G)|^{8k})$.

Step 1: Check whether G contains a jewel or a pyramid (by algorithms in [2]). If it does, output an odd hole and stop. Otherwise, set $\mathcal{F}_1 = \{G\}$ and $\mathcal{F}_2 = \emptyset$.

Step 2: Repeat the following k times. For each graph $F \in \mathcal{F}_1$ and every (P_1, P_2) where $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ are two induced paths of F , add to \mathcal{F}_2 the graph obtained from F by removing the vertex set $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. Set $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{F}_2 = \emptyset$.

Step 3: Set $\mathcal{F} = \mathcal{F}_1$.

Theorem 6 *This algorithm produces the desired output, and its running time is $O(|V(G)|^{8k})$.*

Proof: Suppose that the algorithm does not output an odd hole. Suppose G contains a shortest odd hole H^* . By Step 1 G contains no jewel and no pyramid. Now we show how Step 2 generates a graph in \mathcal{F}_1 that contains H^* and H^* is clean in it.

By Lemma 1, $S(H^*)$ is the set of all H^* -major vertices. Let A be a maximal stable set of $S(H^*)$. We follow the notation in Theorem 5. Let $P_1 = x_0, x_1, x_2, x_3$ and $P_2 = y_0, y_1, y_2, y_3$ such that x_1x_2 and y_1y_2 satisfy the conditions stated in Theorem 5. Let $S'(H^*)$ denote the set of vertices of $S(H^*)$ that have no neighbor in $\{x_1, x_2\}$, and are A -complete. Let G' be the graph obtained from G by removing $(N(x_1) \cup N(x_2) \cup N(y_1) \cup N(y_2)) \setminus (V(P_1) \cup V(P_2))$. Then G' contains H^* and the set of major vertices for H^* in G' is contained in $S'(H^*)$. The clique number of the graph induced by $S'(H^*)$ is one less than the clique number of the graph induced by $S(H^*)$. Hence, by the fact that the clique number of G is bounded by k , Theorem 5 implies that, when the k iterations of Step 2 are completed, some graph $F \in \mathcal{F}_1$ contains H^* and H^* is clean in F . Hence (1) holds.

$O(|V(G)|^{8k})$ graphs are created in Step 2. Hence, (2) holds. The running time of Step 1 is $O(|V(G)|^9)$ as discussed in [2]. The running time of Steps 2 is $O(|V(G)|^{8k})$. Therefore, the overall running time is $O(|V(G)|^{8k})$. \square

In [2] a polynomial time algorithm with following specification is obtained.

Input: A clean graph G .

Output: ODD-HOLE-FREE when G is odd-hole-free, and NOT ODD-HOLE-FREE otherwise.

The above two algorithms imply that it is polynomial to test whether a graph of bounded clique number contains an odd hole.

References

- [1] D. Bienstock, On complexity of testing for odd holes and induced odd paths, *Discrete Mathematics* 90 (1991) 85-92.

[2] M. Chudnovsky, G. Cornu jols, X. Liu, P. Seymour and K. Vu kovi , Recognizing Berge graphs, *Combinatorica* 25 (2005) 143-186.